Nuclear Criticality Safety Engineer Training
Module 4 [1]

Neutron Scattering

LESSON OBJECTIVE

To consider the mechanics of neutron scattering with nuclei and to understand neutron moderation, including moderation through the resonance region.

NEUTRON SCATTERING

In previous modules, two types of neutron scattering were introduced: elastic, in which the momentum and kinetic energy of the system are conserved; and inelastic, in which the momentum is conserved but not the kinetic energy.

The conservation of energy and momentum equations in the laboratory coordinate system for a neutron scattering from an initially stationary nucleus are:

energy $$\frac{1}{2}mv_i^2 = \frac{1}{2}mv_f^2 + \frac{1}{2}MV^2 - Q \quad (1)$$

momentum $$m\vec{v}_i = m\vec{v}_f + M\vec{V} \quad (2)$$

In these equations m is the neutron mass, M the target nucleus mass, v_i is the neutron velocity before the collision, v_f is the neutron velocity after the collision, V is the target nucleus velocity after collision and Q is the net change in the kinetic energy of the system. When generalized to include reactions other than scattering, Q is called the disintegration energy or Q-value of the reaction. For inelastic scattering, the quantity Q is normally retained as energy of excitation of the nucleus after the collision.

The second equation is a vector equation, which could be separated into equations in the x, y and z directions. However, for a single neutron scattering event, conservation of momentum constrains the event to lie in a plane, reducing the momentum equations to a set of two. The three equations (one energy, two momentum) involve seven parameters: two masses, three speeds and two scattering angles.

[1] Developed for the U. S. Department of Energy Nuclear Criticality Safety Program by T. G. Williamson, Ph.D., Westinghouse Safety Management Solutions, Inc., in conjunction with the DOE Criticality Safety Support Group.

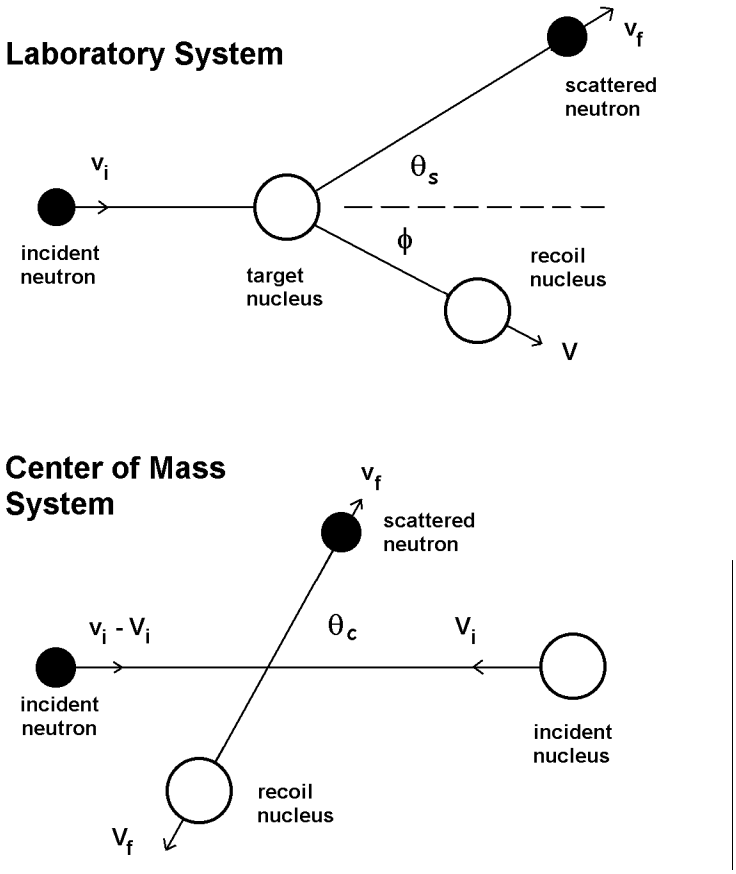


Figure 1. Diagram of Neutron Scattering Event

Looking at the laboratory system in Figure 1 which defines the scattering angles, the two momentum equations are:

$$\begin{aligned} mv_i &= mv_f \cos\theta_s + MV\cos\phi \\ 0 &= mv_f \sin\theta_s - MV\sin\phi \end{aligned} \tag{3}$$

Normally the masses and the neutron incident speed are known and the three equations are arranged to eliminate the speed and scattering angle of the recoil nucleus. After some algebraic manipulations, the energy of the scattered neutron, E_f, can be expressed in terms of the neutron scattering angle, θ_s, and the neutron initial energy, E_i.

$$E_f(\mu_s, E_i) = \frac{1}{(A+1)^2}\left[\mu_s\sqrt{E_i} \pm \sqrt{E_i(\mu_s^2 + A^2 - 1) + A(A+1)Q}\right]^2 \tag{4}$$

where $\mu_s = \cos\theta_s$ and θ_s is the neutron scattering angle in the laboratory coordinate system, and A = M/m, the ratio of the scattering nucleus mass to the neutron mass.

For elastic scattering, Q = 0, and this equation becomes

$$E_f(\mu_s, E_i) = \frac{E_i}{(A+1)^2}\left[\sqrt{A^2 - 1 + \mu_s^2} + \mu_s\right]^2 \tag{5}$$

With this equation the energy of the scattered neutron can be calculated in terms of the initial neutron energy and the scattering angle in the laboratory coordinate system.

We have talked about the laboratory coordinate system but not about a center-of-mass, or center-of-momentum, system. Why might we care about a center-of mass system? Physical measurements are made in the laboratory system, reactors are built in the laboratory system and we do criticality evaluations for systems in the laboratory coordinate system. However, most nuclear physics calculations are more easily done in the center-of-mass system and, for most cases of interest, neutron scattering is isotropic in the center-of-mass system. For low energies, less than a few hundred keV, most of the elastic scattering cross sections are isotropic in the center-of-mass system. For hydrogen, neutron scattering is isotropic in the center-of-mass system to energies well above 20 MeV. For isotropic scattering the angular-dependent scattering cross section in the center-of-mass system is

$$\sigma_s(E, \mu_c) = \frac{\sigma_s(E)}{4\pi} \tag{6}$$

where μ_c is the cosine of the center-of-mass neutron scattering angle. In this equation there is no angular dependence.

The mathematics of the conversion from one system to the other is tedious. The result will be presented in order to learn something about hydrogen scattering. The relations between the differential cross sections in the two coordinate systems are

$$\sigma_s(E, \mu_s) = \sigma_s(E, \mu_c)\frac{\left(1 + 2\gamma\mu_c + \gamma^2\right)^{3/2}}{1 + \gamma\mu_c} \tag{7}$$

and

$$\sigma_s(E, \mu_c) = \sigma_s(E, \mu_s)\frac{\sqrt{1 - \gamma^2(1 - \mu_s^2)}}{\left[\gamma\mu_s \pm \sqrt{1 - \gamma^2(1 - \mu_s^2)}\right]^2} \tag{8}$$

where

$$\frac{1}{\gamma} = \sqrt{A^2 + \frac{A(A+1)Q}{E}} \tag{9}$$

[**NOTE:** Don't get lost in the mathematics of the above equations. The point in presenting them is to show that there are two coordinate systems routinely used to describe neutron interactions and that there are mathematical relations between the two systems. The importance of these relations is in the insight they can provide to the behavior of neutron interactions as described below.]

As stated above, neutron scattering from hydrogen is isotropic in the center-of-mass system. But what does this say about neutron scattering from hydrogen in the laboratory coordinate system, where measurements are made? For hydrogen A = 1, Q = 0 and $\gamma = 1$ and Eqn (8) reduces to

$$\sigma_s(E,\mu_c) = \sigma_s(E,\mu_s)\frac{\mu_s}{(\mu_s \pm \mu_s)^2} \tag{10}$$

Inserting the expression for $\sigma_s(E,\mu_c)$ from Eqn (6) into Eqn (10) and solving for $\sigma_s(E,\mu_s)$, the angular dependent scattering cross section for hydrogen in the laboratory system is

$$\sigma_s(E,\mu_s) = \frac{\sigma_s(E)}{4\pi}4\mu_s = \frac{\sigma_s(E)}{\pi}\cos(\theta_s) \tag{11}$$

Since negative scattering cross sections are not allowed, the hydrogen scattering cross section can be described as:

Center-of-mass	constant at $\sigma_s(E)/4\pi$	
Laboratory	0	for $-1 < \mu_s < 0$ (for $\pi < \theta_s < 2\pi$)
	$\frac{\sigma_s(E)}{4\pi}\mu_s$	for $0 < \mu_s < 1$ (for $0 < \theta_s < \pi$)

Although neutron scattering on hydrogen is isotropic in the center-of-mass system, it is highly anisotropic, i.e., it is all forward-scatter in the laboratory system. The hydrogen laboratory scattering cross section is shown in the following figure. Hydrogen is a special, although a very common, material in criticality evaluations.

For elastic scattering from heavier nuclei, $\gamma = \sqrt{1/A^2}$, and as A becomes large, γ approaches zero. In that case $\sigma_s(E,\mu_s) \rightarrow \sigma_s(E,\mu_c)$, and the distinction between scattering in the two coordinate systems disappears; that is, isotropic scattering in the center-of-mass system is also isotropic in the laboratory system.

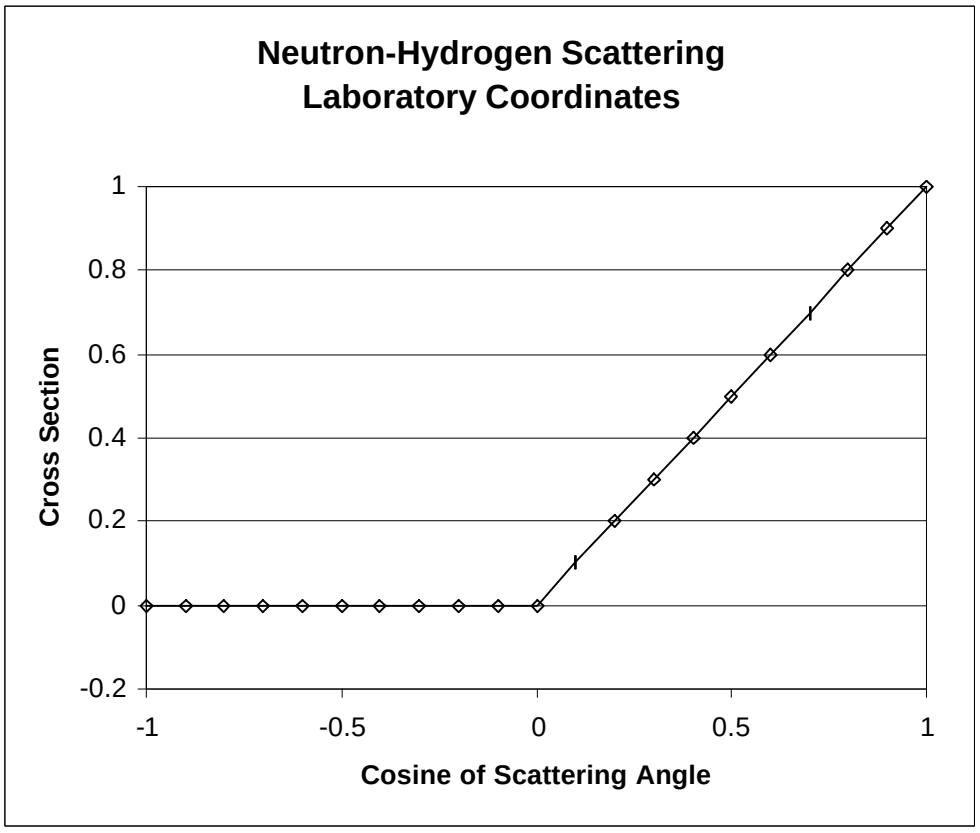


Figure 2. Neutron Scattering from Hydrogen in the Laboratory Coordinate System

Returning to the kinematics, we have relations between the scattering angles in the laboratory and center-of-mass systems and a relation between the scattered neutron energy and the scattering angle in the laboratory system. With some manipulations the equations reduce to

$$E_f = \frac{1}{2}E_i(1+\alpha) + \frac{1}{2}(1-\alpha)E_i\sqrt{1+\frac{Q(A+1)}{E_i A}}\mu_c + Q\frac{A}{A+1} \tag{12}$$

where

$$\alpha \equiv \left(\frac{A-1}{A+1}\right)^2 .$$

For elastic scattering, Q = 0 and Equation 12 reduces to

$$E_f = \frac{1}{2}E_i\left[(1+\alpha) + (1-\alpha)\mu_c\right]. \tag{13}$$

For the case $\mu_c = 1$ (forward scattering), $E_f = E_i$ (i.e., there is no energy loss in a "grazing" collision). For backscatter, $\mu_c = -1$ and $E_f = \alpha E_i$, which is the minimum energy a neutron can have after an elastic scattering collision.

We have derived the maximum and minimum energy loss for a neutron after a scattering collision, but what is the average energy loss? The probability that a neutron with initial energy E_i will have a scattered energy in the energy interval dE_f about E_f is the ratio of the differential cross section to the total cross section.

$$P(E_i \rightarrow E_f)dE_f = \frac{\sigma_s(E_i \rightarrow E_f)dE_f}{\sigma_s(E_i)} \tag{14}$$

The differential energy scattering cross section is related to the differential angular scattering cross section by

$$\sigma_s(E_i \rightarrow E_f)dE_f = \sigma_s(E_i, \mu_c)2\pi d\mu_c \tag{15}$$

In this equation the 2π comes from the azimuthal scattering dependence. Since a single scattering event is constrained to a plane there is no preferential scattering in the azimuthal direction. The differential $dE_f/d\mu_c$ can be calculated from

$$E_f = \frac{1}{2}E_i\left[(1+\alpha) + (1-\alpha)\mu_c\right]. \tag{16}$$

For elastic scattering, which is isotropic in the center-of-mass system, the scattering probability becomes

$$P(E \rightarrow E') = \frac{1}{(1-\alpha)E}$$

The average energy loss is

$$\Delta E_{avg} = \int_{E_{min}}^{E_{max}} \left(E_i - E_f\right)P(E_i \rightarrow E_f)dE_f = \int_{\alpha E_i}^{E_i} \frac{dE_f(E_i - E)_f}{(1-\alpha)E_i}$$

$$\Delta E_{avg} = \frac{1}{2}(1-\alpha)E_i$$

Some values for these scattering parameters for a few materials commonly encountered in criticality situations are listed in the following table. For scattering from hydrogen, on the average, a neutron can lose half of its energy in a single scattering event; for scattering from oxygen about 10% of the energy can be lost, with smaller fractions for scattering from the heavier nuclides of uranium and plutonium.

	A	σ_s	α	$\Delta E_{avg}/E_i$
H	1.0079	20.49	0.00	0.500
C	12.011	4.75	0.72	0.142
O	15.9994	3.76	0.78	0.111
Fe	55.847	11.35	0.93	0.035
^{235}U	235.0439	14.3	0.98	0.008
^{239}Pu	239.0522	7.6	0.98	0.008

Throughout this section we have mentioned elastic and inelastic scattering but have reduced each case to elastic scattering only. Inelastic scattering does not play a significant role in most nuclear criticality considerations. For high energy neutrons, those with energies above 1 MeV, inelastic scattering can be a significant energy loss mechanism. For example, for scattering of high energy neutrons on uranium the energy loss could be near 10% for a single interaction rather than the 0.8% from elastic scattering. Normally, however, the inelastic scattering cross sections are smaller than the elastic and the presence of a little bit of moderator can dominate the energy loss mechanism.

THERMAL NEUTRON SPECTRA

Neutrons are born in fission with energies in the MeV range and interact with nuclei, losing energy with each scattering reaction until the neutron energy is comparable to the energy of thermal motion of the scattering nuclei. At this point the neutron can either gain or lose energy and will be in "thermal equilibrium" with its surroundings. In this case the neutron flux density spectrum is Maxwellian; that is, the energy dependent flux density can be expressed by the Maxwellian distribution

$$\phi(E) = \frac{2\pi n}{(\pi kT)^{3/2}}\left(\frac{2}{m}\right)^{1/2} E e^{-E/kT},$$

where

E is the neutron energy,
n is the density of neutrons,
k is Boltzmann's constant,
m is the neutron mass, and
T is the temperature of the medium in degrees Kelvin.

The energy at which $\phi(E)$ is a maximum, the most probable energy, E_T, is found by placing the derivative of the distribution function equal to zero. It is easily found that

$$E_T = kT$$

and the speed, v_T, of the neutrons having the most probable energy is found from

$$\tfrac{1}{2} m v_T^2 = kT$$

At room temperature (293.7 K), E_T = 0.0253 eV = 1/40 eV and the neutron speed at this energy v_T = 2200 m/s. This is the classic "thermal" neutron. The average energy of the neutron in a Maxwellian distribution is $\frac{3}{2}kT$, about 0.038 eV.

A neutron may be born in a fission event with energy about 2 MeV and end as a thermal neutron with energy 1/40 eV. During the thermalization process the neutron has passed through about 8 decades in energy and traversed the perilous path through nuclear resonances in the intermediate energy range. Figure 3 shows the calculated fission spectrum (see Module 1) and the thermal neutron spectrum, each normalized separately. In the intermediate region the spectrum would be a horizontal straight line in a medium dominated by hydrogen scattering and with negligible resonance absorption.

RESONANCE ABSORPTION AND SCATTERING

The wide variations in nuclear cross sections are caused by discrete energy levels in the nuclear structure. A few general statements about nuclear levels are evident from the cross sections. (See the cross section curves in NCSET Module 2.)

- Higher atomic number nuclides have more nuclear levels that lower atomic weight nuclides.
- Higher atomic weight nuclides have their first resonance at lower energies than lower atomic weight nuclides.
- The lower energy resonances are well separated, the higher energy resonances are closer together and not well resolved.
- The lower energy resonances are larger than the higher energy resonances.
- Absorption resonances are symmetric; scattering resonances are not.

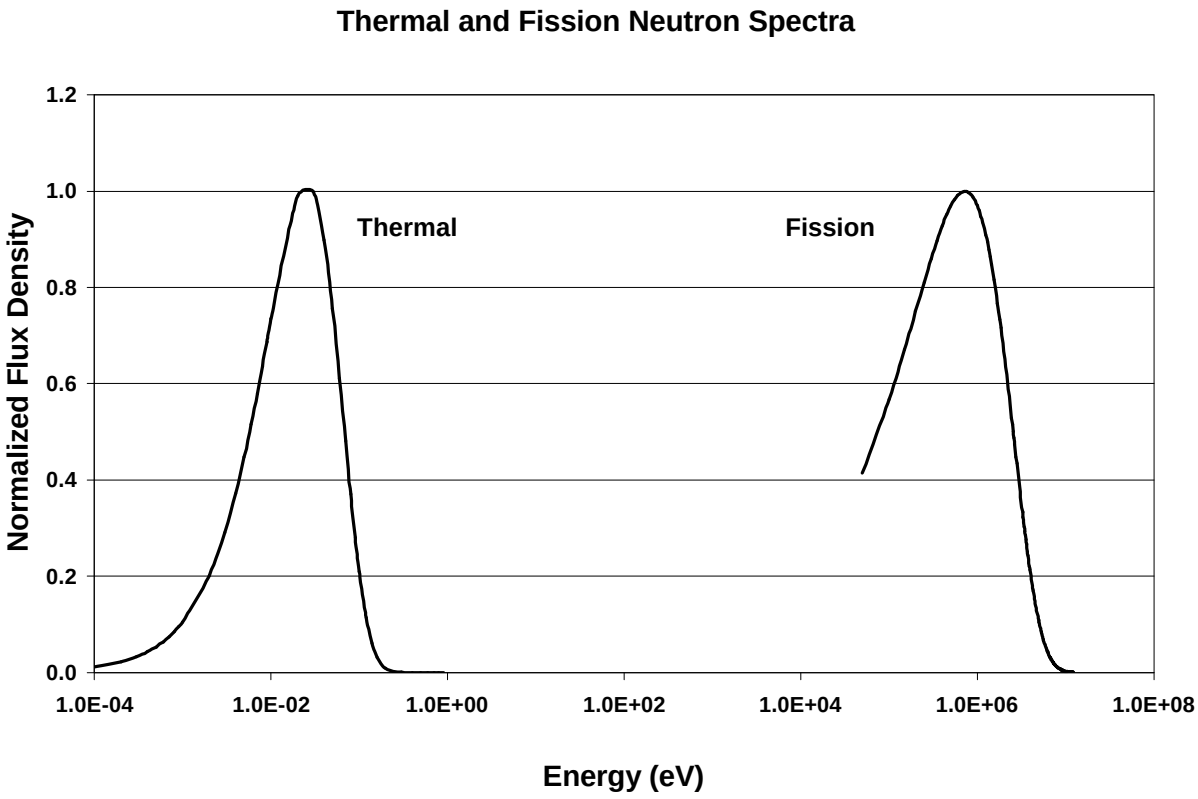


Figure 3. Comparison of Thermal and Fission Neutron Spectra

In this lesson a few equations are presented to illustrate some of these points. For widely spaced resonances, the energy dependence of the absorption cross section can be described by the Breit-Wigner single level resonance formula.

$$\sigma_\gamma\left(E_c\right) = \sigma_0 \frac{\Gamma_\gamma}{\Gamma}\left(\frac{E_0}{E_c}\right)^{\frac{1}{2}} \frac{1}{1+ y^2}$$

with

$$y = \frac{2}{\Gamma}\left(E_c - E_0\right)$$

where

E_0 is the energy of the resonance
E_c is the neutron energy in the center-of-mass system
Γ is the total absorption line width; i.e., the full width at half maximum (FWHM) of the total absorption line width in energy units
Γ_γ is the radiative absorption line width
σ_0 is the value of the total cross section at the resonance energy, E_0.

σ_0 is defined by

$$s_0 = 4p\,\overline{l_0^2}\,\frac{\Gamma_n}{\Gamma}\,g$$

where

g is the nuclear spin factor
λ_0 is the reduced neutron wave length at E_0, and
Γ_n is the neutron line width which is proportional to $E^{1/2}$.

The shape of the resonance is symmetric about E_0 because of y^2 term. The total cross section, σ_0 varies as $1/E_0$.

A scattering resonance shape has three terms:

1) resonance scattering, the Breit-Wigner shape;
2) interference scattering between the incoming and outgoing neutron waves; and
3) potential scattering, i.e., from a hard sphere with no nuclear structure.

These terms are combined in the following equation.

$$\sigma_s(E_c) = \sigma_0\,\frac{\Gamma_n}{\Gamma}\sqrt{\frac{E_0}{E_c}}\,\frac{1}{1+y^2} + \sigma_0\,\frac{2R}{\overline{\lambda_0}}\,\frac{y}{1+y^2} + 4\pi R^2$$

where R is the nuclear radius (given approximately by $1.25 \times 10^{-13}\ A^{1/3}$ cm). The significance of this equation is that the interference term is proportional to y, which can be negative for neutron energies lower than the resonance energy. These are the dips in the cross sections evident for many heavy nuclei. These cross section dips can provide "windows" through which neutrons of specific energies may stream. (See the ^{238}U curves in NCSET Module 2.)

For the nuclear criticality specialist it is important to remember that there is an energy region in which the neutron cross sections may vary widely over narrow energy ranges and that the neutron slowing down through this energy range must be treated properly. Fortunately, this has been done in many of the codes used by the criticality specialist. A few special cases are worthy of mention.

- There are resonances in both the fission and capture cross sections which may not coincide. Thus the energy-dependent capture-to-fission ratio fluctuates in the resonance region.

- ^{238}U has a large absorption resonance at about 6.7 eV which can effect the number of neutrons arriving at thermal energies in low-enriched uranium systems. Lumps of uranium and homogeneous mixtures have different reactivities.

- ^{239}Pu and ^{240}Pu have large overlapping resonances at about 66 eV which can interfere. Lumps of plutonium and homogeneous mixtures have different reactivities.

- Some lighter elements, such as iron and aluminum, may not have all of the resonances properly treated which could lead to errors in unmoderated mixtures of these materials.

SUMMARY

Neutrons lose energy by elastic scattering with nuclei. The lighter the scattering material, the more energy a neutron can lose in a single scattering event. In general, scattering is isotropic in the center-of-mass coordinate system and approaches isotropic in the laboratory system for most nuclides. The exception is scattering from hydrogen which is anisotropic in the laboratory system.

Neutrons are born in fission with energies in the MeV range and interact with nuclei, losing energy with each scattering reaction until the neutron energy is comparable to the energy of thermal motion of the scattering nuclei. During the thermalization process the neutron has passed through about eight decades in energy and through the resonance region in which cross sections may vary widely in magnitude over narrow energy ranges.

PROBLEM

1. Compute and plot the energy of the scattered neutron as a function of the laboratory scattering angle for a 2.0 MeV neutron elastically scattered from hydrogen, oxygen and ^{239}Pu.

PROBLEM SOLUTION

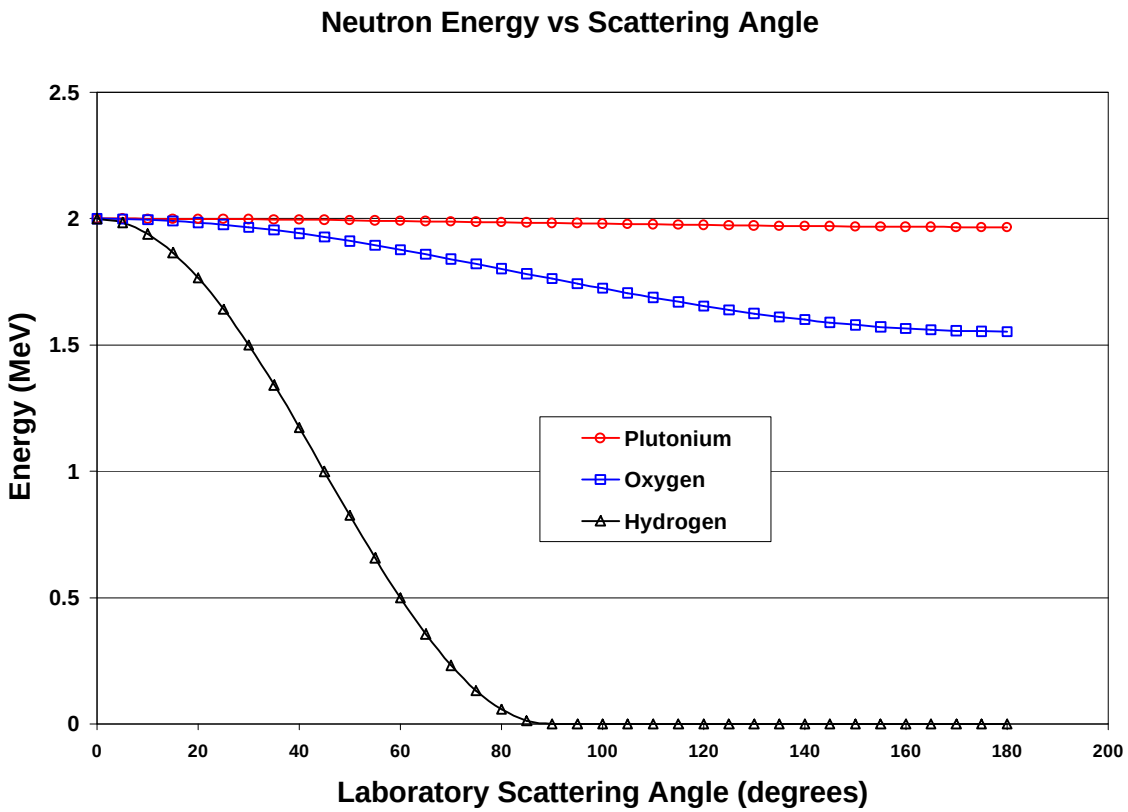
1. From Page 3, the energy of the scattered neutron is given by

$$E_f(\mu_s, E_i) = \frac{E_i}{(A+1)^2}\left[\sqrt{A^2 - 1 + \mu_s^2} + \mu_s\right]^2$$

where $\mu_s = \cos\theta_s$ and θ_s is the neutron scattering angle in the laboratory coordinate system; A = M/m, the ratio of the scattering nucleus mass to the electron mass.

Assume the following mass values:	n	1.0087	H	1.0078
	O	15.9994	^{239}Pu	239.0522

The figure below shows the calculated results. A tabulation of the values is on the following page.



		Scattered Energy		
Angle		H	O	Pu-239
0		2.000	2.000	2.000
5		1.985	1.999	2.000
10		1.940	1.996	2.000
15		1.866	1.991	1.999
20		1.766	1.985	1.999
25		1.642	1.977	1.998
30		1.500	1.966	1.998
35		1.341	1.955	1.997
40		1.173	1.942	1.996
45		0.999	1.927	1.995
50		0.825	1.912	1.994
55		0.657	1.895	1.993
60		0.499	1.878	1.992
65		0.356	1.859	1.990
70		0.232	1.841	1.989
75		0.132	1.821	1.988
80		0.059	1.802	1.986
85		0.013	1.782	1.985
90			1.763	1.983
95		0.0	1.743	1.982
100		0.0	1.725	1.980
105		0.0	1.706	1.979
110		0.0	1.688	1.977
115		0.0	1.671	1.976
120		0.0	1.655	1.975
125		0.0	1.640	1.974
130		0.0	1.625	1.972
135		0.0	1.612	1.971
140		0.0	1.600	1.970
145		0.0	1.590	1.970
150		0.0	1.580	1.969
155		0.0	1.572	1.968
160		0.0	1.566	1.968
165		0.0	1.560	1.967
170		0.0	1.557	1.967
175		0.0	1.554	1.967
180		0.0	1.554	1.967